



# A class of global large solutions to the magnetohydrodynamic equations with fractional dissipation

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**Abstract.** The global existence and regularity problem on the magnetohydrodynamic (MHD) equations with fractional dissipation is not well understood for many ranges of fractional powers. This paper examines this open problem from a different perspective. We construct a class of large solutions to the  $d$ -dimensional ( $d = 2, 3$ ) MHD equations with any fractional power. The process presented here actually assesses that an initial data near any function whose Fourier transform lives in a compact set away from the origin always leads to a unique and global solution.

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## 1. Introduction

The magnetohydrodynamic (MHD) equations govern the motion of electrically conducting fluids such as plasmas, liquid metals and electrolytes (see, e.g., [9, 23]). They are the centerpiece of the magnetohydrodynamics initiated by Alfvén [2]. The MHD equations consist of the Navier–Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism. Due to the nonlinear interaction between the fluid velocity and the magnetic field, the MHD equations can accommodate much richer phenomena than the Navier–Stokes equations alone. One significant example is that the magnetic field can actually stabilize the fluid motion [24].

The MHD equations have always been of great interest in mathematics. Mathematically rigorous foundational work has been laid by Duvaut and Lions [12] and Sermange and Temam [30]. Recently the MHD equations have gained renewed interests and there have been substantial developments on the well-posedness problem, especially when the MHD equations involve only partial or fractional dissipation. A summary on some of the recent results can be found in a review paper [36]. This paper focuses on the 2D and the 3D incompressible MHD equations with fractional dissipation,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^\alpha u = -\nabla P + b \cdot \nabla b, & x \in \mathbb{R}^d, t > 0, \\ \partial_t b + u \cdot \nabla b + \eta(-\Delta)^\beta b = b \cdot \nabla u, & x \in \mathbb{R}^d, t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = u_0(x), & b(x, 0) = b_0(x), \end{cases} \quad (1.1)$$

where  $d = 2$  or  $3$ ,  $u$ ,  $P$  and  $b$  represent the velocity, the pressure and the magnetic field, respectively, and  $\nu > 0$ ,  $\eta > 0$ ,  $\alpha \geq 0$  and  $\beta \geq 0$  are real parameters. The fractional Laplacian operator  $(-\Delta)^\alpha$  is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

The MHD equations with fractional dissipation given by (1.1) have recently attracted considerable interests, due to their mathematical importance and physical applications. Mathematically (1.1) represents a two-parameter family of systems and contains the MHD systems with standard Laplacian dissipation as special cases. (1.1) allows us to simultaneously examine a whole family of equations and potentially reveals how the solution properties are related to the sizes of  $\alpha$  and  $\beta$ . Physically the fractional diffusion operators can model the anomalous diffusion and have now been widely used in turbulence modeling to control the effective range of the non-local dissipation (see, e.g., [1, 16, 17]).

A range of global well-posedness results on (1.1) have been obtained. [34] has shown that (1.1) is globally well posed if  $\alpha$  and  $\beta$  satisfy

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{d}{2}. \quad (1.2)$$

Wu [35] was able to sharpen this result by replacing the fractional Laplacian operators by general Fourier multiplier operators. In particular, (1.1) with a  $\frac{(-\Delta)^\alpha}{\log(I-\Delta)}u$  and  $\frac{(-\Delta)^\beta}{\log(I-\Delta)}b$  for  $\alpha$  and  $\beta$  satisfying (1.2) is also globally well posed. Yamazaki obtained the global regularity for the case when  $\alpha = 2$  and  $\beta = 0$  and for a logarithmically reduced fractional dissipation [40]. A further logarithmic refinement was recently worked out by Yamazaki [41].

More global regularity results beyond those stated above are available for the 2D case. When  $d = 2$ , (1.1) with  $\nu = 0$  and  $\beta > 1$  was shown to always possess global classical solutions by [5] via the Besov space approach and later by [20] via the parabolic regularity estimates. The global regularity for the case when  $\alpha > 0$  and  $\beta = 1$  has also been resolved [13]. A significant improvement of [13] is the global regularity of (1.1) with  $\beta = 1$  and  $(-\Delta)^\alpha u$  replaced by  $\log(I - \Delta)u$  [44]. Discovering and exploring a special structure in the nonlinear terms in the equation of the magnetic field, Dong et al. [10] and [11] were able to sharpen the results of [5, 20] and [13] by removing half of the magnetic diffusion. The critical case  $\nu = 0$  and  $\beta = 1$  has so far resisted a complete resolution. A very recent work establishes the global well posedness with only directional hyperviscosity [42]. Many more exciting results on the global regularity problem are available for the 2D case (see, e.g., [3–5, 10, 11, 13, 19, 20, 32, 33, 37–40, 44]). There is also important progress on the uniqueness of weak local solutions to the MHD equations with partial or fractional dissipation (see, e.g., [6, 8, 14, 15, 18, 26]).

The issue of whether smooth solutions of the MHD equations (1.1) with large initial data can develop singularity in finite time is still a challenging open problem when  $\alpha$  and  $\beta$  are not in the ranges mentioned above. The perspective of this paper is different. Our goal here is to offer an effective approach of constructing large solutions of (1.1). A special consequence of our construction assesses that any initial data close to a function whose Fourier transform supported in a suitable domain away from the origin always leads to a unique global solution of (1.1). We now describe the construction in some detail. There are some differences between the 2D and the 3D cases, so we split our consideration into two cases, one for the 3D case and one for the 2D case, for the sake of clarity. We begin with the 3D case.

To construct large solutions for the 3D MHD equations, we define two suitable vector fields  $\phi_0 \in C_0^\infty(\mathbb{R}^3)$  and  $\psi_0 \in C_0^\infty(\mathbb{R}^3)$  with their Fourier transforms satisfying

$$\widehat{\phi}_0(\xi) = \widehat{\psi}_0(\xi) = \left( \varepsilon^{-1} \log \frac{1}{\varepsilon} \right) \chi(\xi), \quad \xi \in \mathbb{R}^3, \quad (1.3)$$

where  $0 < \varepsilon \leq 1$  is a small parameter depending on  $\nu$  and  $\eta$  and will be specified later. Here,  $\chi$  is a smooth cutoff function,

$$\text{supp } \chi \subset \mathcal{C} \quad \text{and} \quad \chi = 1 \quad \text{on} \quad \mathcal{C}_1,$$

where  $\mathcal{C}$  and  $\mathcal{C}_1$  denote the annuli,

$$\mathcal{C} := \left\{ \xi \in \mathbb{R}^3 \mid |\xi_i - \xi_j| \leq \varepsilon, i, j = 1, 2, 3, 1 \leq |\xi|^2 \leq 2 \right\},$$

$$\mathcal{C}_1 := \left\{ \xi \in \mathbb{R}^3 \mid |\xi_i - \xi_j| \leq \varepsilon, i, j = 1, 2, 3, \frac{5}{4} \leq |\xi|^2 \leq \frac{7}{4} \right\}.$$

We remark that the set  $|\xi_i - \xi_j| \leq \varepsilon$  can be realized by restricting  $\xi \in \mathbb{R}^3$  to the cylinder of radius  $\frac{\varepsilon}{\sqrt{3}}$  centered on the line  $x_1 = x_2 = x_3$ . Suppose  $\xi \in \mathbb{R}^3$  satisfies, for any  $r \in \mathbb{R}$ ,

$$x_1 = x_2 = x_3 = \frac{r}{3}, \quad \xi_1 + \xi_2 + \xi_3 = r, \quad (1.4)$$

$$(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 \leq \frac{1}{3}\varepsilon^2. \quad (1.5)$$

Then, it is not difficult to see that  $|\xi_i - \xi_j| \leq \varepsilon$ ,  $i, j = 1, 2, 3$ . In fact, we have from (1.4) and (1.5)

$$\begin{aligned} & (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 \\ &= \xi_1^2 + \xi_2^2 + \xi_3^2 - 2x_1(\xi_1 + \xi_2 + \xi_3) + 3x_1^2 \\ &= \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\frac{r}{3}r + \frac{r^2}{3} \\ &= \xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{r^2}{3} \leq \frac{1}{3}\varepsilon^2. \end{aligned}$$

Then,

$$\begin{aligned} & (\xi_1 - \xi_2)^2 + (\xi_1 - \xi_3)^2 + (\xi_2 - \xi_3)^2 \\ &= 2(\xi_1^2 + \xi_2^2 + \xi_3^2) - 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) \\ &= 2(\xi_1^2 + \xi_2^2 + \xi_3^2) + (\xi_1^2 + \xi_2^2 + \xi_3^2 - r^2) \\ &\leq \varepsilon^2, \end{aligned}$$

which implies  $|\xi_i - \xi_j| \leq \varepsilon$ ,  $i, j = 1, 2, 3$ . Our global existence and regularity result for the 3D MHD equations can be stated as follows.

**Theorem 1.1.** Assume  $\phi_0$  and  $\psi_0$  are given by (1.3). Define  $U_0$  and  $B_0$  by

$$U_0 = \nabla \times \phi_0, \quad B_0 = \nabla \times \psi_0. \quad (1.6)$$

Let  $\nu > 0$ ,  $\eta > 0$  and  $\alpha \geq 0$  and  $\beta \geq 0$ . Let  $s > \max\{\frac{5}{2} - \alpha, \frac{5}{2} - \beta, 1\}$ . Consider the 3D MHD equations in (1.1) with the initial data

$$u_0 = U_0 + v_0 \quad \text{and} \quad b_0 = B_0 + h_0.$$

If  $\varepsilon$ ,  $v_0 \in H^s(\mathbb{R}^3)$  and  $h_0 \in H^s(\mathbb{R}^3)$  satisfy

$$\varepsilon = C_1 (\min\{\nu, \eta\})^{\frac{10}{9}} \quad \text{and} \quad \|v_0\|_{H^s} + \|h_0\|_{H^s} \leq C_2 \min\{\nu, \eta\}$$

for suitable constants  $C_1 > 0$  and  $C_2 > 0$ , then (1.1) has a unique global solution  $(u, b)$  satisfying

$$u, b \in C([0, \infty); H^s(\mathbb{R}^3)), \quad \Lambda^\alpha u, \Lambda^\beta b \in L^2(0, \infty; L^2(\mathbb{R}^3)).$$

The initial data  $(u_0, b_0)$  in Theorem 1.1 is not small. In fact,

$$\begin{aligned} \|u_0\|_{L^2} &\geq \|U_0\|_{L^2} - \|v_0\|_{L^2} \\ &\geq \left[ \int_{\mathbb{R}^3} |\xi|^2 |\widehat{\phi_0}(\xi)|^2 d\xi \right]^{\frac{1}{2}} - \|v_0\|_{H^s} \end{aligned}$$

$$\begin{aligned}
&\geq \left[ \int_{\mathcal{C}_1} |\xi|^2 |\widehat{\phi}_0(\xi)|^2 d\xi \right]^{\frac{1}{2}} - \|v_0\|_{H^s} \\
&\geq C \left( \varepsilon^{-1} \log \frac{1}{\varepsilon} \right) \varepsilon - \|v_0\|_{H^s} \\
&\geq C \log \frac{1}{\varepsilon} - C_2 \max\{\nu, \eta\},
\end{aligned} \tag{1.7}$$

which can be really large when  $\varepsilon$  is take to be small. Similarly, any homogeneous  $\dot{H}^s$  norm is not small and  $\|b_0\|_{H^s}$  is also large. In addition, as we shall see from the proof, Theorem 1.1 remains valid when the annulus in the definition of  $\mathcal{C}$  is replaced by any compact set supported away from the origin.

A similar result also holds for the 2D MHD equations in (1.1). There are some minor differences in the construction process. We define two scalar functions  $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^2)$  and  $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^2)$  satisfying

$$\widehat{\widetilde{\phi}}(\xi) = \widehat{\widetilde{\psi}}(\xi) = \left( \varepsilon^{-\frac{1}{2}} \log \frac{1}{\varepsilon} \right) \chi_1(\xi), \quad \xi \in \mathbb{R}^2, \tag{1.8}$$

where  $\chi_1$  is a smooth cutoff function,

$$\text{supp} \chi_1 \subset \mathcal{D} \quad \text{and} \quad \chi_1 = 1 \quad \text{on} \quad \mathcal{D}_1.$$

Here,  $\mathcal{D}$  and  $\mathcal{D}_1$  denote the annuli,

$$\begin{aligned}
\mathcal{D} &:= \left\{ \xi \in \mathbb{R}^2 \mid |\xi_i - \xi_j| \leq \varepsilon, i, j = 1, 2, 1 \leq |\xi|^2 \leq 2 \right\}, \\
\mathcal{D}_1 &:= \left\{ \xi \in \mathbb{R}^2 \mid |\xi_i - \xi_j| \leq \varepsilon, i, j = 1, 2, \frac{5}{4} \leq |\xi|^2 \leq \frac{7}{4} \right\}.
\end{aligned}$$

We remark that it is not necessary for  $\widetilde{\phi}$  and  $\widetilde{\psi}$  to be the same. They are taken to be the same here for the sake of brevity. The global existence and regularity result for the 2D MHD equations can be stated as follows.

**Theorem 1.2.** Assume  $\widetilde{\phi}$  and  $\widetilde{\psi}$  are given by (1.8). Define  $\widetilde{U}$  and  $\widetilde{B}$  by

$$\widetilde{U} = \nabla^\perp \widetilde{\phi} := (\partial_2 \widetilde{\phi}, -\partial_1 \widetilde{\phi}), \quad \widetilde{B} = \nabla^\perp \widetilde{\psi}. \tag{1.9}$$

Let  $\nu > 0$ ,  $\eta > 0$  and  $\alpha \geq 0$  and  $\beta \geq 0$ . Let  $s > 2$ . Consider the 2D MHD equations in (1.1) with the initial data

$$u_0 = \widetilde{U} + v_0 \quad \text{and} \quad b_0 = \widetilde{B} + h_0.$$

If  $\varepsilon$ ,  $v_0 \in H^s(\mathbb{R}^2)$  and  $h_0 \in H^s(\mathbb{R}^2)$  satisfy

$$\varepsilon = C_3 (\min\{\nu, \eta\})^{\frac{5}{2}} \quad \text{and} \quad \|v_0\|_{H^s} + \|h_0\|_{H^s} \leq C_4 \min\{\nu, \eta\}$$

for suitable constant  $C_3 > 0$  and  $C_4 > 0$ , then (1.1) has a unique global solution  $(u, b)$  satisfying

$$u, b \in C([0, \infty); H^s(\mathbb{R}^2)), \quad \Lambda^\alpha u, \Lambda^\beta b \in L^2(0, \infty; L^2(\mathbb{R}^2)).$$

Again the initial data  $(u_0, b_0)$  is not small in  $H^s(\mathbb{R}^2)$ . As in (1.7),

$$\|u_0\|_{L^2}, \|b_0\|_{L^2} \geq C \log \frac{1}{\varepsilon} - C_4 \max\{\nu, \eta\}.$$

Theorem 1.2 remains true if the annulus in the definition of  $\mathcal{D}$  is changed to any compact set supported away from the origin.

We mention some related results. Lei et al. [25] constructed smooth large solutions to the 3D Navier–Stokes equations with the initial data close to a Beltrami flow. More information on the Beltrami flow can be found in [7, 29]. Zhou–Zhu [45] obtained a class of large solutions to the 3D damped Euler near the Beltrami flow. Family of large solutions have also been obtained for the damped MHD equations and the

Hall-MHD equations (see [27, 28, 43]). Our construction presented here is somewhat different and does not involve Beltrami flow.

To prove Theorem 1.1, we seek a solution of the form

$$u = U + v, \quad b = B + h,$$

where  $U$  and  $B$  are the solutions of the corresponding linearized equations

$$\begin{cases} \partial_t U + \nu(-\Delta)^\alpha U = 0, \\ \partial_t B + \eta(-\Delta)^\beta B = 0, \\ U(x, 0) = U_0(x), \quad B(x, 0) = B_0(x). \end{cases} \quad (1.10)$$

By the definition of  $U_0$  and  $B_0$  in Theorem 1.1,  $U$  and  $B$  can be written as

$$U(t) = e^{-\nu(-\Delta)^\alpha t} U_0 = e^{-\nu(-\Delta)^\alpha t} \nabla \times \phi_0, \quad B(t) = e^{-\eta(-\Delta)^\beta t} \nabla \times \psi_0. \quad (1.11)$$

The equations of  $(v, h)$  are given by

$$\begin{cases} \partial_t v + u \cdot \nabla v + v \cdot \nabla U + \nu(-\Delta)^\alpha v = -\nabla P + b \cdot \nabla h + h \cdot \nabla B + f, \\ \partial_t h + u \cdot \nabla h + v \cdot \nabla B + \eta(-\Delta)^\beta h = b \cdot \nabla v + h \cdot \nabla U + g, \\ \nabla \cdot v = \nabla \cdot h = 0, \\ v(x, 0) = v_0(x), \quad h(x, 0) = h_0(x), \end{cases} \quad (1.12)$$

where

$$f = -U \cdot \nabla U + B \cdot \nabla B \quad \text{and} \quad g = -U \cdot \nabla B + B \cdot \nabla U. \quad (1.13)$$

Then, it suffices to show that (1.12) has a unique global solution. Since the local well posedness follows from as standard procedure, we focus on the global bound via the bootstrap argument. Details on how to obtain suitable energy inequalities and how the bootstrap argument is applied are given in Sect. 2. The proof of Theorem 1.2 is similar and is sketched in Sect. 3.

The rest of this paper is divided into two sections. Section 2 proves Theorem 1.1, while Sect. 3 provides the proof for Theorem 1.2.

## 2. Proof for Theorem 1.1

This section proves Theorem 1.1. As we described in “Introduction”, it suffices to establish that solutions of (1.12) remain bounded in  $H^s$  for all time. This is verified by deriving suitable energy inequalities and applying the bootstrap argument. As a preparation, we first present some bounds on  $U(t)$  and  $B(t)$  given by (1.11) and  $f$  and  $g$  defined in (1.13).

**Lemma 2.1.** *Let  $\phi_0$  and  $\psi_0$  be given by (1.3),  $U_0$  and  $B_0$  by (1.6),  $U(t)$  and  $B(t)$  by (1.11), and  $f$  and  $g$  by (1.13). Then, the following estimates hold.*

(1) *For any  $\sigma \geq 0$  and  $2 \leq q \leq \infty$ ,*

$$\begin{aligned} \|\Lambda^\sigma \phi_0\|_{L^q(\mathbb{R}^3)}, \|\Lambda^\sigma \psi_0\|_{L^q(\mathbb{R}^3)} &\leq C \varepsilon^{1-\frac{2}{q}} \log \frac{1}{\varepsilon}, \\ \|\Lambda^\sigma U(t)\|_{L^q(\mathbb{R}^3)}, \|\Lambda^\sigma B(t)\|_{L^q(\mathbb{R}^3)} &\leq C \varepsilon^{1-\frac{2}{q}} \log \frac{1}{\varepsilon} e^{-C_0 t}, \end{aligned}$$

*where  $C_0 > 0$  is a constant.*

(2) *For any  $s > \frac{3}{2}$ ,*

$$\|f\|_{H^s} + \|g\|_{H^s} \leq C \varepsilon e^{-2C_0 t} (\|\phi_0\|_{H^{s+2}}^2 + \|\psi_0\|_{H^{s+2}}^2). \quad (2.1)$$

*Proof of Lemma 2.1.* By Hausdorff–Young inequality,

$$\|\Lambda^\sigma \phi_0\|_{L^q(\mathbb{R}^3)} \leq C \|\widehat{\Lambda^\sigma \phi_0}\|_{L^{\tilde{q}}(\mathbb{R}^3)} \leq C \left( \varepsilon^{-1} \log \frac{1}{\varepsilon} \right) \varepsilon^{\frac{2}{q}} = C \varepsilon^{1-\frac{2}{q}} \log \frac{1}{\varepsilon}.$$

Similarly,

$$\begin{aligned} \|\Lambda^\sigma U(t)\|_{L^q(\mathbb{R}^3)} &\leq C \|\widehat{\Lambda^\sigma U}(t)\|_{L^{\tilde{q}}(\mathbb{R}^3)} \\ &\leq C \left( \varepsilon^{-1} \log \frac{1}{\varepsilon} \right) \varepsilon^{\frac{2}{q}} e^{-C_0 t} = C \varepsilon^{1-\frac{2}{q}} \log \frac{1}{\varepsilon} e^{-C_0 t}, \end{aligned}$$

where  $\frac{1}{q} + \frac{1}{\tilde{q}} = 1$ .  $\|\Lambda^\sigma B(t)\|_{L^q(\mathbb{R}^3)}$  obeys the same bound. To prove (2.1), we rewrite the terms in  $f$  and  $g$  so that each term contains a difference  $\partial_i - \partial_j$  with  $i, j = 1, 2, 3$ . Clearly, for  $\phi = e^{-\nu(-\Delta)^{\alpha} t} \phi_0$  and  $\psi = e^{-\eta(-\Delta)^{\beta} t} \psi_0$ ,

$$U = \nabla \times \phi = (\partial_2 \phi_3 - \partial_3 \phi_2, \partial_3 \phi_1 - \partial_1 \phi_3, \partial_1 \phi_2 - \partial_2 \phi_1)$$

and

$$B = \nabla \times \psi = (\partial_2 \psi_3 - \partial_3 \psi_2, \partial_3 \psi_1 - \partial_1 \psi_3, \partial_1 \psi_2 - \partial_2 \psi_1).$$

Then, direct calculations show that the first component of  $-U \cdot \nabla U$  becomes

$$\begin{aligned} &-U \cdot \nabla U^1 \\ &= -\partial_2 \phi_3 \partial_1 \partial_2 \phi_3 + \partial_1 \phi_3 \partial_2 \partial_2 \phi_3 + \partial_2 \phi_3 \partial_1 \partial_3 \phi_2 - \partial_1 \phi_3 \partial_2 \partial_3 \phi_2 \\ &\quad + \partial_3 \phi_2 \partial_1 \partial_2 \phi_3 - \partial_1 \phi_2 \partial_3 \partial_2 \phi_3 - \partial_3 \phi_2 \partial_1 \partial_3 \phi_2 + \partial_1 \phi_2 \partial_3 \partial_3 \phi_2 \\ &\quad - \partial_3 \phi_1 \partial_2 \partial_2 \phi_3 + \partial_2 \phi_1 \partial_3 \partial_2 \phi_3 + \partial_3 \phi_1 \partial_2 \partial_3 \phi_2 - \partial_2 \phi_1 \partial_3 \partial_3 \phi_2 \\ &= [(\partial_1 - \partial_2) \phi_3 \partial_1 \partial_2 \phi_3 + \partial_1 \phi_3 \partial_2 (\partial_2 - \partial_1) \phi_3] \\ &\quad + [(\partial_2 - \partial_1) \phi_3 \partial_1 \partial_3 \phi_2 + \partial_1 \phi_3 \partial_3 (\partial_1 - \partial_2) \phi_2] \\ &\quad + [(\partial_3 - \partial_1) \phi_2 \partial_1 \partial_2 \phi_3 + \partial_1 \phi_2 \partial_2 (\partial_1 - \partial_3) \phi_3] \\ &\quad + [(\partial_1 - \partial_3) \phi_2 \partial_1 \partial_3 \phi_2 + \partial_1 \phi_2 \partial_3 (\partial_3 - \partial_1) \phi_2] \\ &\quad + [(\partial_2 - \partial_3) \phi_1 \partial_2 \partial_2 \phi_3 + \partial_2 \phi_1 \partial_2 (\partial_3 - \partial_2) \phi_3] \\ &\quad + [(\partial_3 - \partial_2) \phi_1 \partial_2 \partial_3 \phi_2 + \partial_2 \phi_1 \partial_3 (\partial_2 - \partial_3) \phi_2]. \end{aligned}$$

Taking the  $H^s$ -norm yields,

$$\begin{aligned} \|-U \cdot \nabla U^1\|_{H^s} &\leq C (\|(\partial_1 - \partial_2) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_2 - \partial_1) \phi\|_{H^{s+1}} \\ &\quad + \|(\partial_2 - \partial_1) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_1 - \partial_2) \phi\|_{H^{s+1}} \\ &\quad + \|(\partial_3 - \partial_1) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_1 - \partial_3) \phi\|_{H^{s+1}} \\ &\quad + \|(\partial_1 - \partial_3) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_3 - \partial_1) \phi\|_{H^{s+1}} \\ &\quad + \|(\partial_2 - \partial_3) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_3 - \partial_2) \phi\|_{H^{s+1}} \\ &\quad + \|(\partial_3 - \partial_2) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_2 - \partial_3) \phi\|_{H^{s+1}}) \\ &\leq C (\|(\partial_i - \partial_j) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_i - \partial_j) \phi\|_{H^{s+1}}), \end{aligned}$$

where  $i, j$  in the last line are summed over  $i, j = 1, 2, 3$ . Similarly,

$$\begin{aligned} \|f\|_{H^s} &\leq C (\|(\partial_i - \partial_j) \phi\|_{H^s} \|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}} \|(\partial_i - \partial_j) \phi\|_{H^{s+1}} \\ &\quad + \|(\partial_i - \partial_j) \psi\|_{H^s} \|\psi\|_{H^{s+2}} + \|\psi\|_{H^{s+1}} \|(\partial_i - \partial_j) \psi\|_{H^{s+1}}) \\ &\leq C e^{-2C_0 t} (\|(\partial_i - \partial_j) \phi_0\|_{H^s} \|\phi_0\|_{H^{s+2}} + \|\phi_0\|_{H^{s+1}} \|(\partial_i - \partial_j) \phi_0\|_{H^{s+1}}) \\ &\quad + C e^{-2C_0 t} (\|(\partial_i - \partial_j) \psi_0\|_{H^s} \|\psi_0\|_{H^{s+2}} + \|\psi_0\|_{H^{s+1}} \|(\partial_i - \partial_j) \psi_0\|_{H^{s+1}}) \end{aligned}$$

$$\begin{aligned}
&\leq C\varepsilon e^{-2C_0 t} (\|\phi_0\|_{H^s} \|\phi_0\|_{H^{s+2}} + \|\phi_0\|_{H^{s+1}} \|\phi_0\|_{H^{s+1}}) \\
&\quad + C\varepsilon e^{-2C_0 t} (\|\psi_0\|_{H^s} \|\psi_0\|_{H^{s+2}} + \|\psi_0\|_{H^{s+1}} \|\psi_0\|_{H^{s+1}}) \\
&\leq C\varepsilon e^{-2C_0 t} (\|\phi_0\|_{H^{s+2}}^2 + \|\psi_0\|_{H^{s+2}}^2).
\end{aligned}$$

The bound for  $\|g\|_{H^s}$  can be similarly obtained. This proves (2.1).  $\square$

In addition, the following commutator and bilinear estimates involving fractional derivatives will be used (see, e.g., [21, 22]).

**Lemma 2.2.** *Let  $s > 0$ . Let  $p, p_1, p_3 \in (1, \infty)$  and  $p_2, p_4 \in [1, \infty]$  satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

*Then, there exist constants  $C$ 's such that*

$$\begin{aligned}
\|[J^s, F]G\|_{L^p} &\leq C (\|J^s F\|_{L^{p_1}} \|G\|_{L^{p_2}} + \|J^{s-1} G\|_{L^{p_3}} \|\nabla f\|_{L^{p_4}}), \\
\|J^s(FG)\|_{L^p} &\leq C (\|J^s F\|_{L^{p_1}} \|G\|_{L^{p_2}} + \|J^s G\|_{L^{p_3}} \|F\|_{L^{p_4}}),
\end{aligned}$$

where  $J = (I - \Delta)^{\frac{1}{2}}$ .

*Proof of Theorem 1.1.* Applying  $J^s$  to (1.12) and dotting with  $(J^s v, J^s h)$  yield

$$\frac{1}{2} \frac{d}{dt} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2) + \nu \|v\|_{H^{s+\alpha}}^2 + \eta \|h\|_{H^{s+\beta}}^2 \leq \sum_{m=1}^7 I_m, \quad (2.2)$$

where, due to  $\nabla \cdot v = 0$ ,  $\nabla \cdot U = 0$ ,  $\nabla \cdot h = 0$  and  $\nabla \cdot B = 0$ ,

$$\begin{aligned}
I_1 &:= - \int [J^s, v \cdot \nabla] v \cdot J^s v \, dx - \int [J^s, v \cdot \nabla] h \cdot J^s h \, dx, \\
I_2 &:= - \int [J^s, U \cdot \nabla] v \cdot J^s v \, dx - \int [J^s, U \cdot \nabla] h \cdot J^s h \, dx, \\
I_3 &:= \int [J^s, h \cdot \nabla] h \cdot J^s v \, dx + \int [J^s, h \cdot \nabla] v \cdot J^s h \, dx, \\
I_4 &:= \int [J^s, B \cdot \nabla] h \cdot J^s v \, dx + \int [J^s, B \cdot \nabla] v \cdot J^s h \, dx, \\
I_5 &:= - \int J^s(v \cdot \nabla U) \cdot J^s v \, dx - \int J^s(v \cdot \nabla B) \cdot J^s h \, dx, \\
I_6 &:= \int J^s(h \cdot \nabla B) \cdot J^s v \, dx + \int J^s(h \cdot \nabla U) \cdot J^s h \, dx, \\
I_7 &:= \int J^s f \cdot J^s v \, dx + \int J^s g \cdot J^s h \, dx.
\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}
|I_1| &\leq C \|\nabla v\|_{L^\infty} \|v\|_{H^s}^2 + C \|\nabla v\|_{L^\infty} \|h\|_{H^s}^2 + C \|\nabla h\|_{L^\infty} \|v\|_{H^s} \|h\|_{H^s} \\
&\leq C \|v\|_{H^s} \|v\|_{H^{s+\alpha}}^2 + C \|v\|_{H^{s+\alpha}} \|h\|_{H^{s+\beta}} \|h\|_{H^s} \\
&\leq C (\|v\|_{H^{s+\alpha}}^2 + \|h\|_{H^{s+\beta}}^2) (\|v\|_{H^s} + \|h\|_{H^s}),
\end{aligned}$$

where we have used the Sobolev inequalities

$$\|\nabla v\|_{L^\infty} \leq C \|v\|_{H^{s+\alpha}} \quad \text{and} \quad \|\nabla h\|_{L^\infty} \leq C \|h\|_{H^{s+\beta}}$$

for  $s > \max\{\frac{5}{2} - \alpha, \frac{5}{2} - \beta\}$ . Applying Lemma 2.2 with  $p_1 < \infty$  large and  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$  and Lemma 2.1,

$$\begin{aligned} |I_2| &\leq C \|v\|_{H^s} (\|J^s U\|_{L^{p_1}} \|\nabla v\|_{L^{p_2}} + \|\nabla U\|_{L^\infty} \|v\|_{H^s}) \\ &\quad + C \|h\|_{H^s} (\|J^s U\|_{L^{p_1}} \|\nabla h\|_{L^{p_2}} + \|\nabla U\|_{L^\infty} \|h\|_{H^s}) \\ &\leq C \varepsilon^{1-\frac{2}{p_1}} \left( \log \frac{1}{\varepsilon} \right) e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2), \end{aligned}$$

where we have used the Sobolev inequality  $\|\nabla v\|_{L^{p_2}} \leq C \|v\|_{H^s}$  for  $s > 1$  and  $p_2 > 2$  but close to 2. For simplicity, we take  $p_1 = 40$  (a concrete number is not crucial here). Using the simple fact that, for  $0 < \varepsilon \leq 1$ ,

$$\varepsilon^{\frac{1}{20}} \left( \log \frac{1}{\varepsilon} \right) \leq C,$$

we have

$$|I_2| \leq C \varepsilon^{\frac{9}{10}} e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2).$$

$I_3$  can be similarly estimated as  $I_1$ ,

$$|I_3| \leq C (\|v\|_{H^{s+\alpha}}^2 + \|h\|_{H^{s+\beta}}^2) (\|v\|_{H^s} + \|h\|_{H^s}).$$

$I_4$  can be similarly estimated as  $I_2$ ,

$$|I_4| \leq C \varepsilon^{\frac{9}{10}} e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2).$$

By Lemma 2.2,

$$\begin{aligned} |I_5| &\leq C \|v\|_{H^s} (\|J^s \nabla U\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|\nabla U\|_{L^\infty} \|v\|_{H^s}) \\ &\quad + C \|v\|_{H^s} (\|J^s \nabla B\|_{L^{p_1}} \|h\|_{L^{p_2}} + \|\nabla B\|_{L^\infty} \|h\|_{H^s}). \end{aligned}$$

Applying Lemma 2.1 and following the estimates for  $I_2$ , we find

$$|I_5| \leq C \varepsilon^{\frac{9}{10}} e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2).$$

$I_6$  can be estimated similarly as  $I_5$ ,

$$|I_6| \leq C \varepsilon^{\frac{9}{10}} e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2).$$

By Lemma 2.1,

$$\begin{aligned} |I_7| &\leq \|f\|_{H^s} \|v\|_{H^s} + \|g\|_{H^s} \|h\|_{H^s} \\ &\leq C \varepsilon e^{-2C_0 t} (\|\phi_0\|_{H^{s+2}}^2 + \|\psi_0\|_{H^{s+2}}^2) \sqrt{\|v\|_{H^s}^2 + \|h\|_{H^s}^2}. \end{aligned}$$

By Lemma 2.1,

$$\|\phi_0\|_{H^{s+2}}, \|\psi_0\|_{H^{s+2}} \leq C \left( \log \frac{1}{\varepsilon} \right).$$

Therefore, if we use the simple fact that, for  $0 < \varepsilon \leq 1$ ,

$$\varepsilon^{\frac{1}{20}} \left( \log \frac{1}{\varepsilon} \right) \leq C,$$

we obtain

$$|I_7| \leq C \varepsilon^{\frac{9}{10}} e^{-2C_0 t} \sqrt{\|v\|_{H^s}^2 + \|h\|_{H^s}^2}.$$



Inserting the estimates in (2.2) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2) + (\nu - C_5 (\|v\|_{H^s} + \|h\|_{H^s})) \|v\|_{H^{s+\alpha}}^2 \\ & \quad + (\eta - C_5 (\|v\|_{H^s} + \|h\|_{H^s})) \|h\|_{H^{s+\beta}}^2 \\ & \leq C_6 \varepsilon^{\frac{9}{10}} e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2) + C_7 \varepsilon^{\frac{9}{10}} e^{-2C_0 t} \sqrt{\|v\|_{H^s}^2 + \|h\|_{H^s}^2}. \end{aligned} \quad (2.3)$$

We apply the bootstrap argument to (2.3) to establish that  $\|v(t)\|_{H^s} + \|h(t)\|_{H^s}$  remains uniform bounded if  $\|v_0\|_{H^s} + \|h_0\|_{H^s}$  is taken to be sufficiently small. The bootstrap argument starts with an ansatz that  $\|v(t)\|_{H^s} + \|h(t)\|_{H^s}$  is bounded, say

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \leq M$$

and shows that  $\|v(t)\|_{H^s} + \|h(t)\|_{H^s}$  actually admits a smaller bound, say

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \leq \frac{1}{2} M$$

when  $\|v_0\|_{H^s} + \|h_0\|_{H^s}$  is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book (see [31, p.21]). To apply the bootstrap argument to (2.3), we assume that

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \leq M := \frac{1}{2C_5} \min\{\nu, \eta\}. \quad (2.4)$$

Clearly, when (2.4) is fulfilled, we have

$$\nu - C_5 (\|v\|_{H^s} + \|h\|_{H^s}) > 0, \quad \eta - C_5 (\|v\|_{H^s} + \|h\|_{H^s}) > 0.$$

It then follows from (2.3) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2) \\ & \leq C_6 \varepsilon^{\frac{9}{10}} e^{-C_0 t} (\|v\|_{H^s}^2 + \|h\|_{H^s}^2) + C_7 \varepsilon^{\frac{9}{10}} e^{-2C_0 t} \sqrt{\|v\|_{H^s}^2 + \|h\|_{H^s}^2} \end{aligned}$$

or

$$\frac{d}{dt} \sqrt{\|v\|_{H^s}^2 + \|h\|_{H^s}^2} \leq C_6 \varepsilon^{\frac{9}{10}} e^{-C_0 t} \sqrt{\|v\|_{H^s}^2 + \|h\|_{H^s}^2} + C_7 \varepsilon^{\frac{9}{10}} e^{-2C_0 t}.$$

By Gronwall's inequality,

$$\begin{aligned} \sqrt{\|v(t)\|_{H^s}^2 + \|h(t)\|_{H^s}^2} & \leq e^{C_6 \varepsilon^{\frac{9}{10}} \int_0^t e^{-C_0 \tau} d\tau} \left( \sqrt{\|v_0\|_{H^s}^2 + \|h_0\|_{H^s}^2} \right. \\ & \quad \left. + \int_0^t C_7 \varepsilon^{\frac{9}{10}} e^{-2C_0 \tau} d\tau \right) \\ & \leq M_1 (\|v_0\|_{H^s} + \|h_0\|_{H^s}) + M_1 \varepsilon^{\frac{9}{10}}, \end{aligned} \quad (2.5)$$

where

$$M_1 = \max \left\{ e^{C_6 C_0^{-1}}, \frac{1}{2} C_7 C_0^{-1} e^{C_6 C_0^{-1}} \right\}.$$

If  $v_0, h_0$  and  $\varepsilon$  satisfy

$$\|v_0\|_{H^s} + \|h_0\|_{H^s} \leq \frac{1}{8\sqrt{2}M_1C_5} \min\{\nu, \eta\}, \quad \varepsilon \leq \left( \frac{\min\{\nu, \eta\}}{8\sqrt{2}M_1C_5} \right)^{\frac{10}{9}}, \quad (2.6)$$

then (2.5) implies

$$\sqrt{\|v(t)\|_{H^s}^2 + \|h(t)\|_{H^s}^2} \leq M_1 \frac{\min\{\nu, \eta\}}{8\sqrt{2}M_1C_5} + M_1 \frac{\min\{\nu, \eta\}}{8\sqrt{2}M_1C_5} = \frac{M}{2\sqrt{2}}.$$

That is,

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \leq \sqrt{2} \sqrt{\|v(t)\|_{H^s}^2 + \|h(t)\|_{H^s}^2} \leq \frac{M}{2}.$$

The bootstrap argument then implies that, for all  $t > 0$ ,

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \leq \frac{M}{2} = \frac{1}{4C_5} \min\{\nu, \eta\}$$

when  $\|v_0\|_{H^s} + \|h_0\|_{H^s}$  and  $\varepsilon$  satisfy (2.6). This proves Theorem 1.1.  $\square$

### 3. Proof of Theorem 1.2

This section proves Theorem 1.2. Since the proof shares many similarities with that for Theorem 1.1, we shall just provide the details for those parts that differ.

As a preparation of the proof, the following lemma provides upper bounds for  $\tilde{U}$ ,  $\tilde{B}$ ,  $f$  and  $g$  in the 2D case.

**Lemma 3.1.** *Let  $\tilde{\phi}$  and  $\tilde{\psi}$  be given by (1.8), and  $\tilde{U}$  and  $\tilde{B}$  by (1.9). Let  $\tilde{U}(t)$  and  $\tilde{B}(t)$  be given by*

$$\tilde{U}(t) = e^{-\nu(-\Delta)^{\alpha}t} \tilde{U}_0, \quad \tilde{B}(t) = e^{-\eta(-\Delta)^{\beta}t} \tilde{B}_0.$$

*Let  $f$  and  $g$  be given by (1.13), namely*

$$f = -\tilde{U} \cdot \nabla \tilde{U} + \tilde{B} \cdot \nabla \tilde{B} \quad \text{and} \quad g = -\tilde{U} \cdot \nabla \tilde{B} + \tilde{B} \cdot \nabla \tilde{U}.$$

*Then, the following estimates hold.*

(1) *For any  $\sigma \geq 0$  and  $2 \leq q \leq \infty$ ,*

$$\begin{aligned} \|\Lambda^{\sigma} \tilde{\phi}\|_{L^q(\mathbb{R}^2)}, \|\Lambda^{\sigma} \tilde{\psi}\|_{L^q(\mathbb{R}^2)} &\leq C \varepsilon^{\frac{1}{2} - \frac{1}{q}} \log \frac{1}{\varepsilon}, \\ \|\Lambda^{\sigma} \tilde{U}(t)\|_{L^q(\mathbb{R}^2)}, \|\Lambda^{\sigma} \tilde{B}(t)\|_{L^q(\mathbb{R}^2)} &\leq C \varepsilon^{\frac{1}{2} - \frac{1}{q}} \log \frac{1}{\varepsilon} e^{-C_0 t}. \end{aligned}$$

(2) *For any  $s > 1$ ,*

$$\|f\|_{H^s} + \|g\|_{H^s} \leq C \varepsilon e^{-2C_0 t} (\|\tilde{\phi}\|_{H^{s+2}}^2 + \|\tilde{\psi}\|_{H^{s+2}}^2). \quad (3.1)$$

*Proof.* The first part of the estimates can be similarly proven as in the proof of Lemma 2.1. To prove (3.1), we write

$$\tilde{\phi} = e^{-\nu(-\Delta)^{\alpha}t} \tilde{\phi}_0, \quad \tilde{\psi} = e^{-\eta(-\Delta)^{\beta}t} \tilde{\psi}_0$$

and rewrite the first component of  $f$  as

$$\begin{aligned} f^1 &= -\tilde{U} \cdot \nabla \tilde{U}^1 + \tilde{B} \cdot \nabla \tilde{B}^1 \\ &= \partial_1 \tilde{\phi} \partial_2 \partial_2 \tilde{\phi} - \partial_2 \tilde{\phi} \partial_1 \partial_2 \tilde{\phi} - \partial_1 \tilde{\psi} \partial_2 \partial_2 \tilde{\psi} + \partial_2 \tilde{\psi} \partial_1 \partial_2 \tilde{\psi} \\ &= (\partial_1 - \partial_2) \tilde{\phi} \partial_2 \partial_2 \tilde{\phi} + \partial_2 \tilde{\phi} \partial_2 (\partial_2 - \partial_1) \tilde{\phi} + (\partial_2 - \partial_1) \tilde{\psi} \partial_2 \partial_2 \tilde{\psi} + \partial_2 \tilde{\psi} \partial_2 (\partial_1 - \partial_2) \tilde{\psi}. \end{aligned}$$

By Hölder's inequality and Sobolev embedding, for  $s > 1$ ,

$$\begin{aligned} \|f^1\|_{H^s} &\leq C (\|(\partial_1 - \partial_2) \tilde{\phi}\|_{H^s} \|\tilde{\phi}\|_{H^{s+2}} + \|\tilde{\phi}\|_{H^{s+1}} \|(\partial_2 - \partial_1) \tilde{\phi}\|_{H^{s+1}} \\ &\quad + \|(\partial_1 - \partial_2) \tilde{\psi}\|_{H^s} \|\tilde{\psi}\|_{H^{s+2}} + \|\tilde{\psi}\|_{H^{s+1}} \|(\partial_2 - \partial_1) \tilde{\psi}\|_{H^{s+1}}) \\ &\leq C e^{-2C_0 t} (\|(\partial_1 - \partial_2) \tilde{\phi}\|_{H^s} \|\tilde{\phi}\|_{H^{s+2}} + \|\tilde{\phi}\|_{H^{s+1}} \|(\partial_2 - \partial_1) \tilde{\phi}\|_{H^{s+1}}) \\ &\quad + C e^{-2C_0 t} (\|(\partial_1 - \partial_2) \tilde{\psi}\|_{H^s} \|\tilde{\psi}\|_{H^{s+2}} + \|\tilde{\psi}\|_{H^{s+1}} \|(\partial_2 - \partial_1) \tilde{\psi}\|_{H^{s+1}}) \end{aligned}$$

$$\begin{aligned}
&\leq C\varepsilon e^{-2C_0 t} (\|\tilde{\phi}\|_{H^s} \|\tilde{\phi}\|_{H^{s+2}} + \|\tilde{\phi}\|_{H^{s+1}} \|\tilde{\phi}\|_{H^{s+1}}) \\
&\quad + C\varepsilon e^{-2C_0 t} (\|\tilde{\psi}\|_{H^s} \|\tilde{\psi}\|_{H^{s+2}} + \|\tilde{\psi}\|_{H^{s+1}} \|\tilde{\psi}\|_{H^{s+1}}) \\
&\leq C\varepsilon e^{-2C_0 t} (\|\tilde{\phi}\|_{H^{s+2}}^2 + \|\tilde{\psi}\|_{H^{s+2}}^2),
\end{aligned}$$

The second component of  $f$  admits the same bound. Therefore,

$$\|f\|_{H^s} \leq \|f^1\|_{H^s} + \|f^2\|_{H^s} \leq C\varepsilon e^{-2C_0 t} (\|\tilde{\phi}\|_{H^{s+2}}^2 + \|\tilde{\psi}\|_{H^{s+2}}^2).$$

$\|g\|_{H^s}$  can be similarly estimated. This completes the proof of Lemma 3.1.  $\square$

The proof of Theorem 1.2 is close to that for Theorem 1.1 and we omit the details.

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